Malaysian Journal of Mathematical Sciences 19(1): 207–222 (2025)
https://doi.org/10.47836/mjms.19.1.11

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

The Distance-Based Topological Indices of the Zero Divisor Graph for Some Commutative Rings with the Calculator App

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Received: 18 April 2024
Accepted: 7 October 2024

Abstract

A topological index is a mathematical expression that applies to any graph that represents a molecular structure. The Harary index, Wiener index and hyper-Wiener index are distance-based topological indices of a graph. These indices use integration values to represent normalised sums of distances from a given vertex to all other vertices in the graph. The vertices represent the zero divisors of a ring, and two vertices are adjacent if their product equals zero. In this paper, these topological indices of the zero divisor graph are computed for some commutative rings $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{pq}$ where $p < q$ are primes and $k \in \mathbb{N}$ by deriving their general formulas. Several examples are provided to illustrate the main theorems. In the end, the calculator app is created by using MATLAB App Designer to compute the edges, vertices and their distance-based topological indices of the zero divisor graph for some commutative rings.

Keywords: topological index; Harary index; zero divisor graph; commutative ring; Wiener index; calculator apps; MATLAB app designer; Hyper-Wiener index.

1 Introduction

A topological index is defined as a topological-invariant quantity that converts a molecular graph to a mathematical real number. Topological indices are useful in biochemistry, chemistry, and nanotechnology for analysing structure-property relationships, structure-activity relationships, and designing pharmaceuticals [12]. In theoretical chemistry, distance-based topological indices are used to show the physical, biological, and other properties of chemicals. The topological descriptors of a graph are most closely associated with its degrees, distances, and eccentricities. In graph theory, a graph denoted as $\Gamma = (V, E)$, consists of a non-empty set of vertices V and a set of edges E.

This paper focuses on the distance-based topological indices. The Wiener index, introduced by a chemist, Harold Wiener in 1947 [25], shed light on the relationship between molecular structure and physical properties, particularly paraffin boiling points. Pirzada et al. [16] determined the size, degree of the vertices, automorphism group, dimension, and Wiener index of the zero divisor graph of $\mathbb{Z}_{p^n}$. The Wiener index of the zero-divisor graph of the ring of Gaussian integers over $\mathbb{Z}_n$ has been discussed by Selvakumar et al. [20]. Heydari [11] determined lower and upper bounds for the Wiener index and the hyper-Wiener index of the Kragujevac trees. Khalid and Idrees [12] computed the Wiener and the hyper-Wiener indices of Dutch windmill graph and derived their general formulas. In [7], Feng and Liu provided precise lower and upper bounds for the hyper-Wiener index of graphs based on size, order, and diameter. Egan et al. [6] investigated the Harary and Wiener indices of the S-splitting graph for some families of graph. Asir and Rabikka in [4] presented a constructed method to calculate the Wiener index of zero-divisor graph of $\mathbb{Z}_n$ for any positive integer n. Gowtham and Husin [9] examined the reverse topological indices, namely the reverse Zagreb index, the reverse arithmetic-geometric, the geometric-arithmetic, the reverse Nirmala indices for the bistar graphs. The Harary index and the hyper-Wiener index of the ideal-based zero divisor graph of a ring have been calculated by Balamoorthy et al. in [5].

In the field of graph theory, a zero divisor graph of a ring R denoted as $\Gamma(R)$, has been selected for the purpose of this study. Anderson and Livingston [2] proposed and studied $\Gamma(R)$ with zero divisors as its vertices. Anderson and McClurkin in [3] considered generalizations of $\Gamma(R)$ by modifying the vertices or adjacency relations of $\Gamma(R)$. Magi et al. [14] explored the characteristic polynomial and the eigenvalues of the zero divisor graph of a class of commutative rings. Meanwhile, Wei and Luo [24] described the structure of the zero divisor graph and the compressed zero divisor graph of the ring of integer modulo $p^s q^t$ for all distinct primes p, q and $s, t \in \mathbb{N}$ by partition of the vertex set. Additionally, Rehman et al. [22, 23] explored various topological indices over the weakly zero divisor graph of the ring $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ where p, q and r are prime numbers greater than 2. The non-zero divisor graphs of commutative ring, which is the ring of integers modulo n where $n = 8k$ and $k \leq 3$ have been determined by Zai et al. [26]. The development and implementation of algorithms in MAPLE for constructing zero divisor graphs, aiming to identify their capabilities, constraints, and computational efficiency have been discovered by Ali [1].

In this article, the Wiener index, the hyper-Wiener index and the Harary index of $\Gamma(R)$ for some commutative rings $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{pq}$ where $p < q$ are primes and $k \in \mathbb{N}$ are computed.

2 Preliminaries and Known Results

In this section, we provide some definitions in ring theory, graph theory, and topological indices. For this study, some propositions and theorems are also presented from the previous research.

Definition 2.1. [18] *For all $a, b \in R$, a ring R is commutative if and only if $ab = ba$.*

Definition 2.2. [8] *Two nonzero elements of a finite ring R, denoted as a and b, in which $ab = 0$, constitute the zero divisors of R.*

Definition 2.3. [2] *The zero divisor graph of a commutative ring with identity, $\Gamma(R)$, is a simple graph of R with vertices are a set of zero divisors in R, and two distinct vertices a and b are adjacent if and only if $ab = 0$ or $ba = 0$.*

Proposition 2.1. [10] *For any graph Γ, the number of edges,*

$$|E(\Gamma)| = \frac{1}{2} \sum_{a \in V(\Gamma)} deg(a),$$

where $V(\Gamma)$ is the vertex set of Γ and $deg(a)$ represents the number of edges connected to vertex a.

Proposition 2.2. [19] *Let M and N be finite sets. Then, $|M \cup N| = |M| + |N| - |M \cap N|$.*

Consider Γ to be a simple connected graph for Definitions 2.4, 2.5 and 2.6. Meanwhile for Theorems 2.1, 2.2 and 2.3, let Γ be a simple connected graph with diam $(\Gamma) \leq 2$.

Definition 2.4. [25] *The Wiener index of a graph Γ,*

$$W(\Gamma) = \sum_{x,y \in V(\Gamma)} d(x,y),$$

where $d(x,y)$ is the distance between vertices x and y in Γ.

Theorem 2.1. [15],

$$W(\Gamma) = (|V(\Gamma)| - 1)|V(\Gamma)| - |E(\Gamma)|.$$

Definition 2.5. [13] *The hyper-Wiener index of Γ,*

$$WW(\Gamma) = \frac{1}{2}\left(\sum_{x,y \in V(\Gamma)} d(x,y) + \sum_{x,y \in V(\Gamma)} d(x,y)^2\right) = \frac{1}{2}\left(W(\Gamma) + \sum_{x,y \in V(\Gamma)} d(x,y)^2\right).$$

Theorem 2.2. [15],

$$WW(\Gamma) = \frac{3}{2}(|V(\Gamma)| - 1)|V(\Gamma)| - 2|E(\Gamma)|.$$

Definition 2.6. [17] *The Harary index of Γ,*

$$H(\Gamma) = \sum_{x,y \in V(\Gamma)} \frac{1}{d(x,y)}.$$

Theorem 2.3. [15],

$$H(\Gamma) = \frac{1}{4}(|V(\Gamma)| - 1)|V(\Gamma)| + \frac{1}{2}|E(\Gamma)|.$$

The following propositions are the known results of $\Gamma(R)$ for some commutative rings $\mathbb{Z}_{p^k}$.

Proposition 2.3. [21] *The set of zero divisors in* $\mathbb{Z}_{p^k}$ *is given by* $\left\{p\Big(1,2,3,4,\ldots,\left(p^{k-1}-1\right)\Big)\right\}$ *for* p *is prime and* k *is a positive integer.*

The case where $k = 1$ is addressed in Proposition 2.4, while the case where $p = 2$ and $k = 2$ are addressed in Proposition 2.5. Proposition 2.6 addresses the remaining cases.

Proposition 2.4. [21] $\Gamma\left(\mathbb{Z}_p\right)$ *has zero edges.*

Proposition 2.5. [21] $\Gamma\left(\mathbb{Z}_4\right)$ *has zero edges.*

Proposition 2.6. [21] *The number of edges for* $\Gamma\left(\mathbb{Z}_{p^k}\right)$*,*

$$\left|E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right| = \frac{1}{2}\left[(k-1)\left(p^k - p^{k-1}\right) - p^{k-1} - p^{\left\lceil \frac{k-1}{2}\right\rceil} + 2\right].$$

3 Main Results

This section describes some results related to three types of the distance-based topological indices of the zero divisor graph for some commutative rings, namely the Wiener index, the hyper-Wiener index and the Harary index. We include the zero divisors for some commutative rings $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{pq}$ in this section.

3.1 The distance-based topological indices for $\Gamma\left(\mathbb{Z}_{p^k}\right)$

We continue our discussion in this subsection by concentrating on the commutative ring $\mathbb{Z}_{p^k}$, where p is a prime and $k \in \mathbb{N}$. From the previous results in [21], the following proposition and theorems are stated for the distance-based topological indices for $\Gamma\left(\mathbb{Z}_{p^k}\right)$.

Proposition 3.1. *The number of zero divisors in the commutative ring* $\mathbb{Z}_{p^k}$, $\left|Z\left(\mathbb{Z}_{p^k}\right)\right| = p^{k-1} - 1$.

Proof. By Proposition 2.3 with the cardinality, $\left|Z\left(\mathbb{Z}_{p^k}\right)\right| = p^{k-1} - 1$. □

Note that, $\left|Z\left(\mathbb{Z}_{p^k}\right)\right| = \left|V\left(\mathbb{Z}_{p^k}\right)\right| = p^{k-1} - 1$.

In [4], the authors determined the Wiener index of $\Gamma\left(\mathbb{Z}_{p^k}\right)$ where $k \geq 2$ and $p^k \neq 4$ as a theorem that defines the set of vertices and then describes the conditions under which edges are formed between these vertices. Using the results in [15, 21], we then let $\Gamma\left(\mathbb{Z}_{p^k}\right)$ be a simple connected graph with $\operatorname{diam}\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) \leq 2$ for Theorems 3.1, 3.2 and 3.3 in finding their distance-based topological indices of a graph.

Theorem 3.1. *The Wiener index of* $\Gamma\left(\mathbb{Z}_{p^k}\right)$ *is given by,*

$$W\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = \left(p^{k-1}-1\right)\left(p^{k-1}-2\right) - \frac{1}{2}\left[(k-1)\left(p^k - p^{k-1}\right) - p^{k-1} - p^{\left\lceil \frac{k-1}{2}\right\rceil} + 2\right].$$

Proof. By Theorem 2.1, Propositions 2.6 and 3.1, we have

$$\begin{aligned} W\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &= \left(\left|V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right|-1\right)\left|V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right| - \left|E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right| \\ &= \left(\left(p^{k-1}-1\right)-1\right)\left(p^{k-1}-1\right) - \frac{1}{2}\left[(k-1)\left(p^k-p^{k-1}\right)-p^{k-1}-p^{\lceil\frac{k-1}{2}\rceil}+2\right] \\ &= \left(p^{k-1}-2\right)\left(p^{k-1}-1\right) - \frac{1}{2}\left[(k-1)\left(p^k-p^{k-1}\right)-p^{k-1}-p^{\lceil\frac{k-1}{2}\rceil}+2\right]. \end{aligned}$$

□

Theorem 3.2. *The hyper-Wiener index of* $\Gamma\left(\mathbb{Z}_{p^k}\right)$ *is shown by,*

$$WW\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = \frac{3}{2}\left(p^{k-1}-1\right)\left(p^{k-1}-2\right)-(k-1)\left(p^k-p^{k-1}\right)+p^{k-1}+p^{\lceil\frac{k-1}{2}\rceil}-2.$$

Proof. By Theorem 2.2, Propositions 2.6 and 3.1, we get

$$\begin{aligned} WW\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &= \frac{3}{2}\left(\left|V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right|-1\right)\left|V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right| - 2\left|E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right| \\ &= \frac{3}{2}\left(\left(p^{k-1}-1\right)-1\right)\left(p^{k-1}-1\right) - \frac{2}{2}\left[(k-1)\left(p^k-p^{k-1}\right)-p^{k-1}-p^{\lceil\frac{k-1}{2}\rceil}+2\right] \\ &= \frac{3}{2}\left(p^{k-1}-2\right)\left(p^{k-1}-1\right)-(k-1)\left(p^k-p^{k-1}\right)+p^{k-1}+p^{\lceil\frac{k-1}{2}\rceil}-2. \end{aligned}$$

□

Theorem 3.3. *The Harary index of* $\Gamma\left(\mathbb{Z}_{p^k}\right)$ *is presented by,*

$$H\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = \frac{1}{4}\left[\left(p^{k-1}-1\right)\left(p^{k-1}-2\right)+(k-1)\left(p^k-p^{k-1}\right)-p^{k-1}-p^{\lceil\frac{k-1}{2}\rceil}+2\right].$$

Proof. By Theorem 2.3, Propositions 2.6 and 3.1, we obtain

$$\begin{aligned} H\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &= \frac{1}{4}\left(\left|V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right|-1\right)\left|V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right| + \frac{1}{2}\left|E\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)\right| \\ &= \frac{1}{4}\left(\left(p^{k-1}-1\right)-1\right)\left(p^{k-1}-1\right) + \frac{1}{2\cdot 2}\left[(k-1)\left(p^k-p^{k-1}\right)-p^{k-1}-p^{\lceil\frac{k-1}{2}\rceil}+2\right] \\ &= \frac{1}{4}\left[\left(p^{k-1}-2\right)\left(p^{k-1}-1\right)+(k-1)\left(p^k-p^{k-1}\right)-p^{k-1}-p^{\lceil\frac{k-1}{2}\rceil}+2\right]. \end{aligned}$$

□

Example 3.1. *Figure 1 shows* $\Gamma\left(\mathbb{Z}_{32}\right)$ *when* $p=2$ *and* $k=5$.

By using Theorem 3.1, we have

$$W\left(\Gamma\left(\mathbb{Z}_{32}\right)\right) = \left(2^{5-1}-1\right)\left(2^{5-1}-2\right)-\frac{1}{2}\left[(5-1)\left(2^5-2^{5-1}\right)-2^{5-1}-2^{\lceil\frac{5-1}{2}\rceil}+2\right] = 187.$$

By using Theorem 3.2, we get

$$WW\left(\Gamma\left(\mathbb{Z}_{32}\right)\right) = \frac{3}{2}\left(2^{5-1}-1\right)\left(2^{5-1}-2\right)-(5-1)\left(2^5-2^{5-1}\right)+2^{5-1}+2^{\lceil\frac{5-1}{2}\rceil}-2 = 269.$$

By using Theorem 3.3, we obtain

$$H\left(\Gamma\left(\mathbb{Z}_{32}\right)\right) = \frac{1}{4}\left[\left(2^{5-1}-1\right)\left(2^{5-1}-2\right)+(5-1)\left(2^5-2^{5-1}\right)-2^{5-1}-2^{\lceil\frac{5-1}{2}\rceil}+2\right] = 64.$$

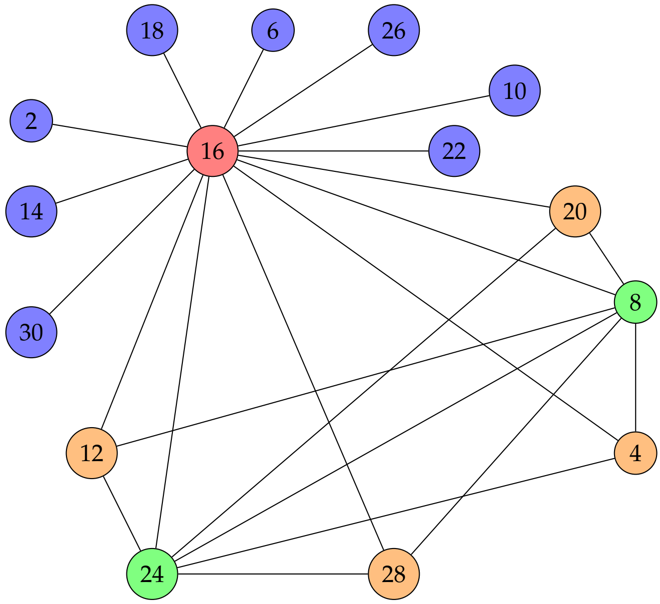


Figure 1: $\Gamma(\mathbb{Z}_{32})$.

3.2 The distance-based topological indices for $\Gamma(\mathbb{Z}_{pq})$

For the rest of this discussion in this subsection, we focus on the commutative ring $\mathbb{Z}_{pq}$, for positive integers k, and $p < q$ are primes. Some results related to $\Gamma(\mathbb{Z}_{pq})$, are described in the following propositions.

Proposition 3.2. *The set of all zero divisors of $Z(\mathbb{Z}_{pq})$ is given by,*

$$Z(\mathbb{Z}_{pq}) = \{p, 2p, 3p, 4p, \ldots, p(q-1)\} \cup \{q, 2q, 3q, 4q, \ldots, q(p-1)\}.$$

Proof. Let $a \in Z(\mathbb{Z}_{pq})$,

(a) Suppose $a \in \mathbb{Z}_{pq}$ with $\gcd(a, p) > 1$ and M is $Z(\mathbb{Z}_{pq})$ for a.
Then, $M = \{p, 2p, 3p, 4p, \ldots, p(q-1)\}$ with the cardinality $(q-1)$.

(b) Suppose $a \in \mathbb{Z}_{pq}$ with $\gcd(a, q) > 1$ and N is $Z(\mathbb{Z}_{pq})$ for a.
Then, $N = \{q, 2q, 3q, 4q, \ldots, q(p-1)\}$ with the cardinality $(p-1)$.

(c) Suppose $a \in \mathbb{Z}_{pq}$ with $\gcd(a, pq) = p$ and $\gcd(a, pq) = q$ or $M \cap N$ is the set of all zero divisors of a.
Then, $M \cap N = \{\}$ with the cardinality 0.

Therefore, $Z(\mathbb{Z}_{pq}) = M \cup N = \{p, 2p, 3p, 4p, \ldots, p(q-1)\} \cup \{q, 2q, 3q, 4q, \ldots, q(p-1)\}$. □

Proposition 3.3. $|Z(\mathbb{Z}_{pq})| = p - 2 + q$.

Proof. By using Proposition 2.2 and Proposition 3.2 with their cardinalities, we have

$$|Z(\mathbb{Z}_{pq})| = |M| + |N| - |M \cap N| = (q-1) + (p-1) - 0 = p - 2 + q.$$

□

Note that, $|Z(\mathbb{Z}_{pq})| = |V(\mathbb{Z}_{pq})| = p - 2 + q$.

Proposition 3.4. *Let $a \in Z(\mathbb{Z}_{pq})$ with $gcd(a, pq) = p$. Then, $deg(a) = p - 1$.*

Proof. Given $a \in Z(\mathbb{Z}_{pq})$ with $\gcd(a, pq) = p$ and given $b \in Z(\mathbb{Z}_{pq})$ with $\gcd(b, pq) = q$ where a and b are adjacent to each other. Since $\gcd(b, pq) = q$, so $b \in q\mathbb{Z}_{pq}$ and

$$|q\mathbb{Z}_{pq}| = |q\{0, 1, 2, 3, ..., pq - 1\}| = \frac{pq}{q} - 1.$$

Thus, since $0 \notin Z(\mathbb{Z}_{pq})$, so $deg(a) = p - 1$. □

Proposition 3.5. *Let $a \in Z(\mathbb{Z}_{pq})$ with $gcd(a, pq) = q$. Then, $deg(a) = q - 1$.*

Proof. Given $a \in Z(\mathbb{Z}_{pq})$ with $gcd(a, pq) = q$, and given $b \in Z(\mathbb{Z}_{pq})$ with $\gcd(b, pq) = p$ where a and b are adjacent to each other. Since $\gcd(b, pq) = p$, so $b \in p\mathbb{Z}_{pq}$ and

$$|p\mathbb{Z}_{pq}| = |p\{0, 1, 2, 3, ..., pq - 1\}| = \frac{pq}{p} - 1.$$

Thus, $deg(a) = q - 1$. □

Proposition 3.6. *Let $a \in V(\Gamma(\mathbb{Z}_{pq}))$, thus $a \in Z(\mathbb{Z}_{pq})$ where $gcd(a, pq) = p$. Then,*

$$|V(\Gamma(\mathbb{Z}_{pq}))| = q - 1.$$

Proof. Given $a \in Z(\mathbb{Z}_{pq})$ where gcd $(a, pq) = p$. Then, $|V(\Gamma(\mathbb{Z}_{pq}))| = 1$ and $b \in Z(\mathbb{Z}_{pq})$ where $\gcd(b, q) = q$ then $|V(\Gamma(\mathbb{Z}_{pq}))| = q - 1$. Using the concept of $|\mathbb{Z}_{pq}| = |\mathbb{Z}_p| \cdot |\mathbb{Z}_q|$, so

$$|V(\Gamma(\mathbb{Z}_{pq}))| = (1)(q - 1) = q - 1.$$

□

Proposition 3.7. *Let $a \in V(\Gamma(\mathbb{Z}_{pq}))$, thus $a \in Z(\mathbb{Z}_{pq})$ where $gcd(a, pq) = q$. Then,*

$$|V(\Gamma(\mathbb{Z}_{pq}))| = p - 1.$$

Proof. Given $a \in Z(\mathbb{Z}_{pq})$ where $\gcd(a, pq) = q$. Then $|V(\Gamma(\mathbb{Z}_{pq}))| = p - 1$ and $b \in Z(\mathbb{Z}_{pq})$ where $\gcd(b, q) = 1$ then $|V(\Gamma(\mathbb{Z}_{pq}))| = 1$. Using the concept of $|\mathbb{Z}_{pq}| = |\mathbb{Z}_p| \cdot |\mathbb{Z}_q|$, so

$$|V(\Gamma(\mathbb{Z}_{pq}))| = (p - 1)(1) = p - 1.$$

□

Proposition 3.8. *The number of edges for $\Gamma(\mathbb{Z}_{pq})$, $|E(\Gamma(\mathbb{Z}_{pq}))| = (q - 1)(p - 1)$.*

Proof. Using Propositions 2.1, 3.4, 3.5, 3.6, and 3.7, we have

$$|E(\Gamma(\mathbb{Z}_{pq}))| = \frac{1}{2}\Big[(p - 1)(q - 1) + (q - 1)(p - 1)\Big].$$

After the simplification of the equation, $|E(\Gamma(\mathbb{Z}_{pq}))| = (q - 1)(p - 1)$. □

Then, theorems are developed for the distance-based topological indices of $\Gamma(\mathbb{Z}_{pq})$. For Theorems 3.4, 3.5 and 3.6, let $\Gamma(\mathbb{Z}_{pq})$ be a simple connected graph with diam $(\Gamma(\mathbb{Z}_{pq})) \leq 2$.

Theorem 3.4. *The Wiener index of $\Gamma(\mathbb{Z}_{pq})$ is given by, $W(\Gamma(\mathbb{Z}_{pq})) = p^2 + q^2 + pq - 4p - 4q + 5$.*

Proof. By Theorem 2.1, Propositions 3.3 and 3.8, we have

$$\begin{aligned} W(\Gamma(\mathbb{Z}_{pq})) &= (|V(\Gamma(\mathbb{Z}_{pq}))| - 1)|V(\Gamma(\mathbb{Z}_{pq}))| - |E(\Gamma(\mathbb{Z}_{pq}))| \\ &= ((p-2+q)-1)(p-2+q) - (q-1)(p-1) \\ &= p^2 + q^2 + pq - 4(p+q) + 5. \end{aligned}$$

□

Theorem 3.5. *The hyper-Wiener index of $\Gamma(\mathbb{Z}_{pq})$ is shown by,*

$$WW(\Gamma(\mathbb{Z}_{pq})) = \frac{3}{2}\left(p^2+q^2\right) - \frac{11}{2}(p+q) + pq + 7.$$

Proof. By Theorem 2.2, Propositions 3.3 and 3.8, we get

$$\begin{aligned} WW(\Gamma(\mathbb{Z}_{pq})) &= \frac{3}{2}(|V(\Gamma(\mathbb{Z}_{pq}))| - 1)|V(\Gamma(\mathbb{Z}_{pq}))| - 2|E(\Gamma(\mathbb{Z}_{pq}))| \\ &= \frac{3}{2}((p-2+q)-1)(p-2+q) - 2(q-1)(p-1) \\ &= \frac{3}{2}\left(p^2+q^2\right) - \frac{11}{2}(p+q) + pq + 7. \end{aligned}$$

□

Theorem 3.6. The Harary index of $\Gamma(\mathbb{Z}_{pq})$ is presented by,

$$H(\Gamma(\mathbb{Z}_{pq})) = \frac{1}{4}\left(p^2+q^2\right) - \frac{7}{4}(p+q) + pq + 2.$$

Proof. By Theorem 2.3, Propositions 3.3 and 3.8, we obtain

$$\begin{aligned} H(\Gamma(\mathbb{Z}_{pq})) &= \frac{1}{4}(|V(\Gamma(\mathbb{Z}_{pq}))| - 1)|V(\Gamma(\mathbb{Z}_{pq}))| + \frac{1}{2}|E(\Gamma(\mathbb{Z}_{pq}))| \\ &= \frac{1}{4}((p-2+q)-1)(p-2+q) + \frac{1}{2}(q-1)(p-1) \\ &= \frac{1}{4}\left(p^2+q^2\right) - \frac{7}{4}(p+q) + pq + 2. \end{aligned}$$

□

Example 3.2. *When $p = 3$ and $q = 11$, $\Gamma(\mathbb{Z}_{33})$ is illustrated in Figure 2.*

By Theorem 3.4, we have

$$W(\Gamma(\mathbb{Z}_{33})) = 3^2 + 11^2 + 3(11) - 4(3+11) + 5 = 112.$$

By Theorem 3.5, we get

$$WW(\Gamma(\mathbb{Z}_{33})) = \frac{3}{2}\left(3^2+11^2\right) - \frac{11}{2}(3+11) + 3(11) + 7 = 158.$$

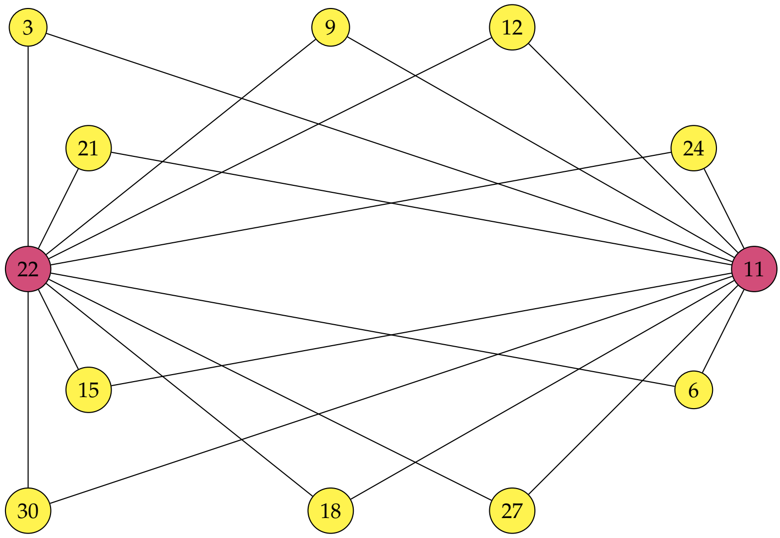


Figure 2: $\Gamma\left(\mathbb{Z}_{33}\right)$

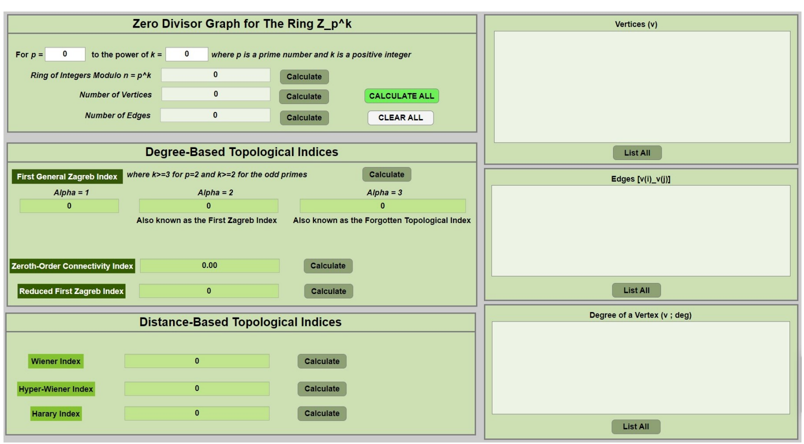
By Theorem 3.6, we obtain

$$H\left(\Gamma\left(\mathbb{Z}_{33}\right)\right)=\frac{1}{4}\left(3^2+11^2\right)-\frac{7}{4}(3+11)+3(11)+2=43.$$

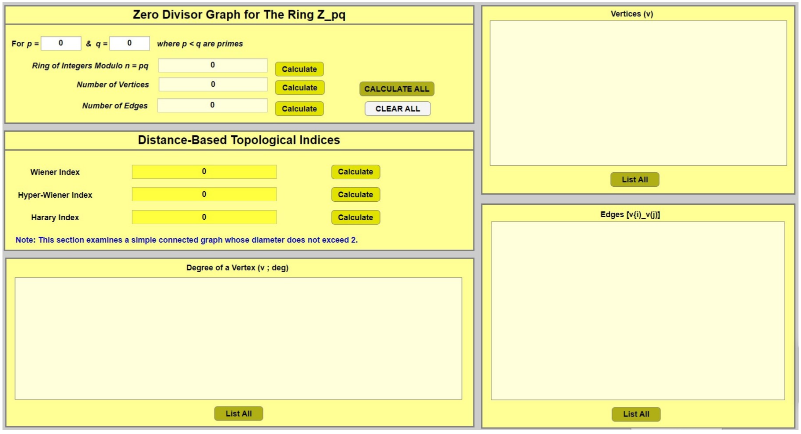
Thus, the answers from the preceding examples demonstrate that the definitions and theorems produce the same results.

4 The Calculator App by Using MATLAB App Designer

In this section, the calculator app for the distance-based topological indices of the zero divisor graph for some commutative rings are described and displayed the interface in Figure 3. We developed this calculator app using MATLAB App Designer to facilitate the creation of a user-friendly interface and efficient computational functionalities.



(a) The calculator app for the distance-based topological indices of $\Gamma\left(\mathbb{Z}_{p^k}\right)$.



(b) The calculator app for the distance-based topological indices of $\Gamma\left(\mathbb{Z}_{pq}\right)$.

Figure 3: The calculator app features a user-friendly interface with interactive buttons and displays.

In this section, the MATLAB programming codes and then the outputs are displayed . In Subsection 3.2, we present $\Gamma\left(\mathbb{Z}_{33}\right)$ as an example. Firstly, the number of vertices and edges of $\Gamma\left(\mathbb{Z}_{33}\right)$ by entering the required values such that $p = 3$ and $q = 11$.

```
function pEditField_6ValueChanged(app, event)
    value = app.pEditField_6.Value;
    if ~isprime(value)
    errordlg('p must be a prime number. Please enter a prime number
        .', 'Invalid Input', 'modal');
    % Reset the value of p to 0
    app.pEditField_6.Value = 0;
    end
end
function qEditField_5ValueChanged2(app, event)
    value = app.qEditField_5.Value;
    if ~isprime(value)
        errordlg('q must be a prime number. Please enter a prime
            number.', 'Invalid Input', 'modal');
    % Reset the value of q to 0
    app.qEditField_5.Value = 0;
    end
end
function CalculateButton_43Pushed(app, event)
    p = app.pEditField_6.Value;
    q = app.qEditField_5.Value;
    if p >= q
        errordlg('p must be less than q. Please enter valid values.
            ', 'Invalid Input', 'modal');
    return;
    end
    %Display the results in the text area
    app.RingofIntegersModulonpqEditField_3.Value = p * q;
    app.NumberofVerticesEditField_6.Value = q + p - 2;
```

```
    app.NumberofEdgesEditField_6.Value = (p - 1) * (q-1);
end
```

According to Figure 2, the output from the display is the same.

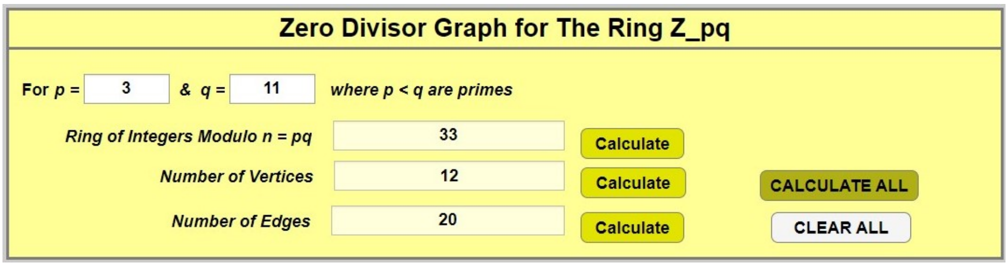


Figure 4: The number of vertices and edges for $\Gamma\left(\mathbb{Z}_{33}\right)$.

In addition, the error messages and prompts are appeared in the following if the user inputs invalid data.

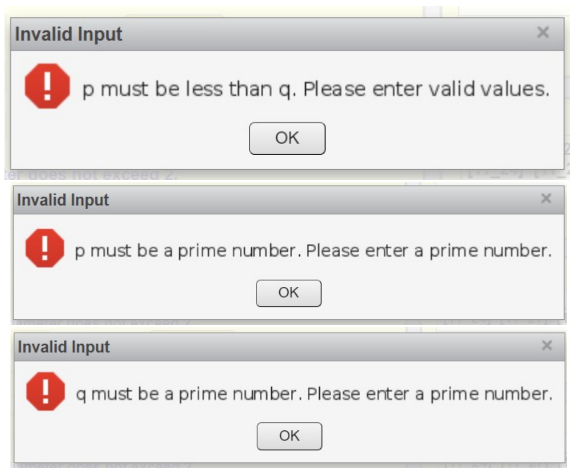


Figure 5: Displays error messages and provides the user with prompts.

The vertices of $\Gamma\left(\mathbb{Z}_{33}\right)$ are listed as follows:

```
function ListAllButton_14Pushed(app, event)
    E1 = p:p:p*(q-1);
    E2 = q:q:q*(p-1);
    result_set_6 = union(E1, E2);
    %Display the vertices in the text area
    app.VerticesvTextArea_5.Value = num2str(result_set_6);
end
```

The output on the screen is shown as follows:

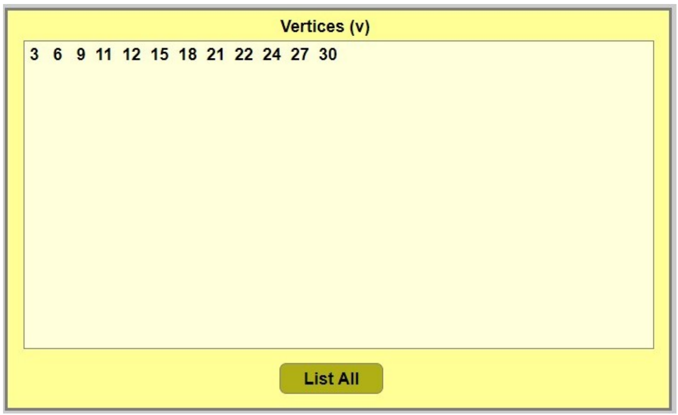


Figure 6: The set of vertices.

The edges of $\Gamma(\mathbb{Z}_{33})$ are presented as follows:

```
function ListAllButton_15Pushed(app, event)
edge_list_6 = [];
% Loop through all pairs of vertices
for i = 1:numel(result_set_6)
    for j = i+1:numel(result_set_6)
        % Check if the product of vertices i and j is a zero divisor
        if mod(result_set_6(i) * result_set_6(j), (p*q)) == 0
            % If it is, add the edge to the edge list
            edge_list_6 = [edge_list_6; result_set_6(i), result_set_6(j)];
        end
    end
end
edge_str_6 = '';
for i = 1:size(edge_list_6, 1)
    edge_str_6 = [edge_str_6, sprintf(' [%d_%d] ', edge_list_6(i, 1), edge_list_6(i, 2))];
end
% Display the edge list in the text area
app.Edgesvi_vjTextArea_5.Value = edge_str_6;
end
```

The display output:

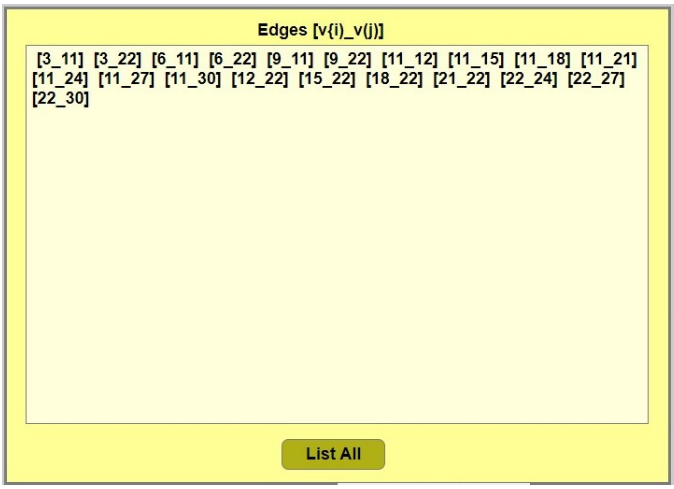


Figure 7: The set of edges.

The degree of a vertex for $\Gamma(\mathbb{Z}_{33})$ are shown as follows:

```
function ListAllButton_16Pushed(app, event)
degree_list_6 = zeros(size(result_set_6));
% Loop through all pairs of vertices
for i = 1:numel(result_set_6)
    for j = i+1:numel(result_set_6)
        % Check if the product of vertices i and j is a zero divisor
        if mod(result_set_6(i) * result_set_6(j), (p*q)) == 0
            % If it is, add the edge to the edge list
            edge_list_6 = [edge_list_6; result_set_6(i), result_set_6(j)];
          % Increment the degrees of the incident vertices
            degree_list_6(i) = degree_list_6(i) + 1;
            degree_list_6(j) = degree_list_6(j) + 1;
        end
    end
end
vertex_degree_str_6 = '';
for i = 1:numel(result_set_6)
    vertex_degree_str_6 = [vertex_degree_str_6, sprintf(' (%d ; %d) ', result_set_6 (i), degree_list_6(i))];
end
% Display the vertex degree in the text area
app.DegreeofaVertexvdegTextArea_5.Value = vertex_degree_str_6;
end
```

The display output:

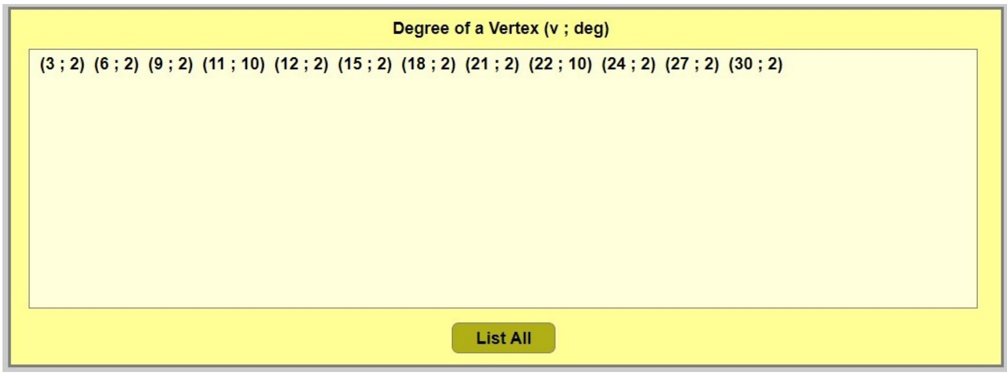


Figure 8: The degree of a vertex.

The distance-based topological indices of $\Gamma(\mathbb{Z}_{33})$ are computed as follows:

```
function CalculateButton_47Pushed(app, event)
   % Perform the calculation only if p < q
    app.WienerIndexEditField_3.Value = (p + q - 2) * (p + q - 3) -
        (q - 1) * (p - 1);
    app.HyperWienerIndexEditField_3.Value =  3/2 * ((p + q - 2) * (
        p + q - 3)) - 2 * (q - 1) * (p - 1);
    app.HararyIndexEditField_3.Value = 1/4 * (p + q - 2) * (p + q -
         3) + 1/2 * (q - 1) * (p - 1);
end
```

The display output corresponds precisely to Example 3.2.

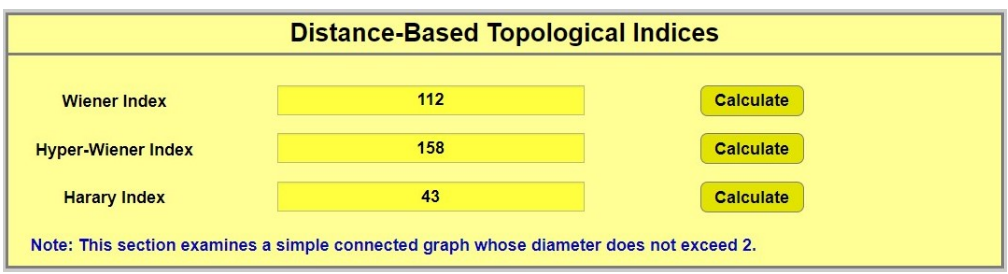


Figure 9: The distance-based topological indices.

Therefore, we can use the calculator app to compute the distance-based topological indices of:

(a) $\Gamma\left(\mathbb{Z}_{p^k}\right)$ for any prime value of p and $k \in \mathbb{N}$.

(b) $\Gamma(\mathbb{Z}_{pq})$ for any prime value of p and q where $p < q$.

5 Conclusion

The general formulas for the Harary index, the Wiener index and the hyper-Wiener index of the zero divisor graph for some commutative rings are computed for these three types of distance-based topological indices. Furthermore, the calculator app is developed using MATLAB App Designer to compute these topological indices of $\Gamma\left(\mathbb{Z}_{p^k}\right)$ and $\Gamma\left(\mathbb{Z}_{pq}\right)$.

Acknowledgement The authors express their gratitude to the Ministry of Higher Education Malaysia (MoHE) for providing financial support to this project through the Fundamental Research Grant Scheme (FRGS/1/2020/STG06/UTM/01/2). Also, the first author would like to thank Universiti Teknologi MARA (UiTM) for giving him study leave.

Conflicts of Interest All authors have stated that they have no competing interests.

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